Lesson 6. Second Order Dynamical Systems

1 Linear second order dynamical systems

- A **second order dynamical system** is a DS of the form

$$A_{n+2} = f(A_{n+1}, A_n) \quad n = 0, 1, 2, \ldots$$

- In other words, in a second order DS,

- The initial conditions of a second order DS specify

- A **linear second order DS** is a DS of the form

$$A_{n+2} = aA_{n+1} + bA_n + c \quad n = 0, 1, 2, \ldots \tag{$\star$}$$

- A linear second order DS always has solutions

Example 1. Consider the DS $A_{n+2} = -A_{n+1} + 6A_n$, $n = 0, 1, 2, \ldots$ with the $A_0 = 7$, $A_1 = -6$. Find the first next five terms of the sequence: A_2, A_3, A_4, A_5, A_6.

2 Finding solutions to a linear second order DS

- Find the roots r, s of the **characteristic equation**

$$x^2 = ax + b$$

- If $a + b \neq 1$, then the general solution to ($\star$) is

$$A_n = \begin{cases} c_1 r^n + c_2 s^n + \frac{c}{1-a-b} & \text{if } r \neq s \\ (c_1 + c_2 n)r^n + \frac{c}{1-a-b} & \text{if } r = s \end{cases} \qquad \text{for any values of } c_1, c_2$$

- If $a + b = 1$, then the general solution to ($\star$) is

$$A_n = \begin{cases} c_1(a-1)^n + c_2 + \left(\frac{c}{2-a}\right)n & \text{if } a + b = 1, a \neq 2 \\ c_1 + c_2 n + \left(\frac{c}{2}\right)n^2 & \text{if } a = 2, b = -1 \end{cases} \qquad \text{for any values of } c_1, c_2$$

- Note that r and s could be imaginary! We will not consider examples of this type

Example 2. Consider the DS $A_{n+2} = -A_{n+1} + 6A_n$, $n = 0, 1, 2, \ldots$.

    a. Find the general solution to this DS.
    b. Find the particular solution to this DS that satsifies the IC $A_0 = 7$, $A_1 = -6$.
    c. Does your answer to b match your answer to Example 1?

Example 3. Find the particular solution to the DS $A_{n+2} = 6A_{n+1} - 9A_n + 2$ that satisfies $A_0 = 1$, $A_1 = 1$. What is A_{10}?

Example 4. Find the particular solution to the DS $A_{n+2} = 3A_{n+1} - 2A_n + 5$ that satisfies $A_0 = 1$, $A_1 = 0$. What is A_{10}?

Example 5. Find the particular solution to the DS $A_{n+2} = 2A_{n+1} - A_n + 3$ that satisfies $A_0 = 0$, $A_1 = -1$. What is A_{10}?

Example 6. Find the particular solution to the DS $A_{n+2} = 2A_{n+1} - A_n + 4$ that satisfies $A_0 = 3$, $A_1 = 6$. What is A_{10}?